

WILL

Relational Orbital Mechanics (R.O.M.)

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Abstract

R.O.M. does not describe how a body moves under forces; it classifies the algebraically allowed relational states of a bound system. It is not equations of motion but algebraically closed system of allowed states within WILL - allowed free will.

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1 Closed Algebraical System of R.O.M. Equations

Desmos Project

R.O.M.

Remark 1.1. *R.O.M. does not describe how a body moves under forces; it classifies the algebraically allowed relational states of a bound system. It is not equations of motion but algebraically closed system of allowed states within WILL - allowed free will.*

$$\kappa^2 = 1 - \frac{1}{(1 + z_\kappa)^2} \quad (z_\kappa = \text{gravitational redshift})$$

$$\beta^2 = 1 - \frac{1}{(1 + z_\beta)^2} \quad (z_\beta = \text{transverse Doppler shift})$$

Observational Inputs

$$Z_{sys} = (1 + z_\kappa)(1 + z_\beta) = \frac{1}{\tau} \quad (\text{product of gravitational red shift and transverse Doppler shift})$$

$$\tau = \kappa_X \cdot \beta_Y = \frac{1}{Z_{sys}} \quad (\text{product of projectinal phase factors on } S^1 \text{ and } S^2 \text{ carriers})$$

$$z_\kappa = \frac{1}{\kappa_X} - 1 \quad (\text{gravitational redshift})$$

$$z_\beta = \frac{1}{\beta_Y} - 1 \quad (\text{transverse Doppler shift})$$

$$z_{\kappa o} = \frac{1}{\kappa_{Xo}} - 1 \quad (\text{redshift at phase } o)$$

$$z_{\beta o} = \frac{1}{\beta_{Yo}} - 1 \quad (\text{transverse Doppler shift at phase } o)$$

Global System Parameters (Fixed for the Orbit)

$$\kappa = \sqrt{\frac{R_s}{a}} = \sqrt{\frac{\rho}{\rho_{max}}} = \sqrt{\kappa_p^2(1 - e)} = \sqrt{4W} = \sqrt{2(\kappa_o(o)^2 - \beta_o(o)^2)} = \sqrt{\frac{1}{2}(3 - \sqrt{1 + 8\tau W_o(O_o)^2})} = \sqrt{1 - (1 + z_\kappa)^{-2}} =$$

$$\kappa_R^2 \left(\frac{I_t}{T}\right)^{\frac{2}{3}} = \kappa_o^2 \cdot \eta_o \quad (\text{potential projection at semi-major axis})$$

$$\kappa_X = \cos(\theta_2) = \sqrt{1 - \kappa^2} = \frac{1}{(1 + z_\kappa)} \quad (\text{potential phase factor})$$

$$\kappa_R = \sqrt{\frac{\kappa^2}{\sin(\theta_R)}} = \sqrt{\frac{R_s}{\frac{T}{2\pi} c \beta \sin(\theta_R)}} \quad (\text{potential projection at the surface})$$

$$\beta = \frac{\kappa}{\sqrt{2}} = \beta_p \sqrt{\frac{1 - e}{1 + e}} = \sqrt{2W} = \frac{4\pi^2 a^2}{T^2 c^2} = \left(\frac{R_s}{\frac{T}{\pi} c}\right)^{\frac{2}{3}} = \sqrt{(\kappa_o(o)^2 - \beta_o(o)^2)} = \sqrt{\frac{\kappa_p^2}{2}(1 - e)} = \sqrt{\kappa_o^2 - \frac{\kappa_o^2}{2} \cdot \left(1 + \left(\frac{1}{\delta_o(o)} - 1\right)\right)} =$$

$$\beta_o(o) \frac{\sqrt{1 - e^2}}{\sqrt{1 + e^2 + 2e \cos(o)}} = \sqrt{1 - (1 + z_\beta)^{-2}} \quad (\text{kinetic projection on semi major axis})$$

$$\beta_Y = \sin(\theta_1) = \sqrt{1 - \beta^2} \quad (\text{kinetic phase factor})$$

$$\beta_{int} = \frac{\beta}{\sqrt{1 - e^2}} \quad (\text{interior kinetic projection})$$

$$\tau = \kappa_X \beta_Y \quad (\text{relational spacetime factor})$$

$$\tau_Y = \sqrt{1 - \tau^2} = \sqrt{3\beta^2 - 2\beta^4} \quad (\text{relational spacetime phase factor})$$

$$Q = \sqrt{\kappa^2 + \beta^2} = \sqrt{\frac{3}{2}} \kappa = \sqrt{3} \beta \quad (\text{total relational shift (magnitude of state difference)})$$

$$R_s = \kappa_o(o)^2 r_o(o) = \frac{2Gm_0}{c^2} = \frac{\Delta t_o(o)}{\zeta(o)} \kappa^2 \beta c = T c \frac{\beta^3}{\pi} = t_o(o) \kappa_o(o)^2 \cdot c = \frac{r_1 r_2}{r_2 - r_1} (\beta_1^2 - \beta_2^2) = \frac{a}{2} (3 - \sqrt{1 + 8\tau W_o(O_o)^2}) =$$

$$= \frac{r_o(o)}{2(2a - r_o(o))} (4a - r_o(o) - \sqrt{(4a - r_o(o))^2 - 8a(2a - r_o(o))(1 - \tau W_o(o)^2)}) = \frac{T}{\pi} \left(\sqrt{1 - \left(\frac{1}{1 + z_\beta}\right)^2}\right)^3 c \quad (\text{Schwarzschild radius - system scale})$$

$$a = \frac{R_s}{\kappa^2} = \frac{R_s}{4W} = \frac{\kappa c T}{2\pi \sqrt{2}} = \frac{\beta_o c}{\omega} \cdot \frac{\sqrt{1 - e^2}}{\sqrt{1 + e^2 + 2e \cos(o)}} = \frac{\Delta t_o(o)}{\zeta} \beta c \quad (\text{semi-major axis})$$

$$R_{surf} = a \cdot \theta_R \quad (\text{physical radius})$$

$$m_0 = \frac{\kappa^2 c^2 a}{2G} = 4\pi \rho a^3 \quad (\text{mass parameter})$$

$$t = \frac{a}{c} \quad (\text{temporal scale of the system})$$

$$W = \frac{\beta^2}{2} = \frac{1}{2}(\kappa_o^2 - \beta_o^2) = \frac{1}{4}\kappa_p^2(1 - e) = \frac{1}{2}(\kappa_o(o)^2 - \frac{\kappa_o(o)^2}{2\delta_o(o)}) = \frac{1}{2}((1 - (1 + z_{\kappa o}(o))^{-2}) - (1 - (1 + z_{\beta o}(o))^{-2})) = \left(\frac{\pi R_s}{2\sqrt{2}cT_o}\right)^{\frac{2}{3}}$$

(energy invariant - binding energy)

$$\Delta\phi = \frac{2\pi \cdot \tau^2}{1 - e^2} = \frac{2\pi(3\beta^2 - 2\beta^4)}{1 - e^2} \quad (\text{precession of perihelion per orbit})$$

$$h_W = r_o(o) \beta_T(o) c = r_o^2 \cdot \omega_o = a \cdot \beta c \cdot e_Y = R_s c \frac{\sqrt{1 - e^2}}{2\beta} \quad (\text{invariant angular momentum})$$

$$\omega = \frac{\beta c}{a} \quad (\text{angular frequency})$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi a}{\beta c} = \frac{2\pi r_o(O_o)}{\beta_o(O_o)c} = \frac{\pi R_s}{c\beta^3} = \frac{\pi R_s}{2\sqrt{2}c} W^{-\frac{3}{2}} \quad (\text{orbital period})$$

Eccentricity Relations

$$e = \frac{1}{\delta_p} - 1 = 1 - \frac{2\beta_a^2}{\kappa_a^2} = \frac{2\beta_p^2}{\kappa_p^2} - 1 \text{ (eccentricity derived from closure)}$$

$$e_Y = \sqrt{1 - e^2} \text{ (eccentricity's orthogonal value)}$$

$$e_X = \frac{1+e}{1-e} = \frac{r_a}{r_p} = \frac{\delta_a}{\delta_p} = \frac{\beta_p}{\beta_a} = \frac{\kappa_p^2}{\kappa_a^2} = \frac{\kappa_a^2 \beta_p^2}{\kappa_p^2 \beta_a^2} \text{ (shape factor, still looks like magic to me)}$$

Time-phase

$$\omega_o(o) = \frac{\beta \cdot c}{a} \cdot \frac{(1+e \cdot \cos(o))^2}{(1-e^2)^{\frac{3}{2}}} \text{ (angular frequency at phase } o)$$

$$\Delta_{to}(o) = \int_0^o \frac{1}{\omega_\theta(\theta)} d\theta = \frac{a}{\beta c} \zeta = \frac{T_O}{2\pi} \zeta \text{ (time duration of given phase interval)}$$

$$\zeta = (1 - e^2)^{\frac{3}{2}} \cdot \left(\int_0^o (1 + e \cdot \cos(\theta))^{-2} d\theta \right) = \frac{1}{\sqrt{1-e^2}} \int_0^o \left(\frac{t_o(\theta)}{t} \right)^2 d\theta \text{ (Temporal Phase interval)}$$

Perihelion Relations

$$r_p = a(1 - e) = \frac{R_s}{\kappa_p^2} \text{ (radius at perihelion)}$$

$$\kappa_p = \kappa \sqrt{\frac{1}{1-e}} = \sqrt{\frac{2\beta^2}{1-e}} = Q_p \sqrt{\frac{2}{3+e}} \text{ (potential projection at perihelion)}$$

$$\kappa_{Xp} = \sqrt{1 - \kappa_p^2} \text{ (potential phase projection at perihelion)} \quad \beta_p = \frac{V_p}{c} = \beta \sqrt{\frac{1+e}{1-e}} = \frac{\kappa_p}{\sqrt{2\delta_p}} = \frac{\kappa_p}{\sqrt{2}} \sqrt{1+e} \text{ (kinetic projection at perihelion)}$$

$$\delta_p = \frac{\kappa_p^2}{2\beta_p^2} = \frac{1}{1+e} \text{ (closure factor at perihelion)}$$

$$Q_p = \sqrt{\kappa_p^2 + \beta_p^2} \text{ (relational shift at perihelion)}$$

Aphelion Relations

$$r_a = a(1 + e) = \frac{R_s}{\kappa_a^2} \text{ (radius at aphelion)}$$

$$\beta_a = \sqrt{\beta_p^2 e_X^{-2}} = \beta \sqrt{e_X^{-1}} \text{ (kinetic projection at aphelion)}$$

$$\kappa_a = \sqrt{2W + \beta_a^2} \text{ (potential projection at aphelion)}$$

$$\delta_a = \frac{1}{1-e} = \frac{\kappa_a^2}{2\beta_a^2} \text{ (closure factor at aphelion)}$$

$$Q_a = \sqrt{\kappa_a^2 + \beta_a^2} \text{ (relational shift at aphelion)}$$

Phase Variables (Depend on o)

o = orbital phase in radians

$$r = r_o(o) = a \frac{1-e^2}{1+e \cos o} = \frac{R_s}{\kappa_o^2} \text{ (radial distance at phase } o)$$

$$\kappa_o = \sqrt{\beta_R(o)^2 + \beta_T(o)^2 + \beta^2} = \sqrt{\frac{R_s}{r}} = 1 - (1 + z_{\kappa o}(o))^{-2} = \kappa_p \sqrt{\frac{1+e \cos(o)}{1+e}} = \sqrt{\beta^2 + \beta_o^2} = \sqrt{\kappa_{surface}^2 \cdot \sin(\theta_{obs})} \text{ (local potential at phase } o)$$

$$\kappa_{Xo} = \sqrt{1 - \kappa_o^2} = (z_{\kappa o}(o) + 1)^{-1} \text{ (gravitational phase factor at phase } o)$$

$$\beta_o(o) = \beta \cdot \frac{\sqrt{1+e^2+2 \cdot e \cdot \cos(o)}}{\sqrt{1-e^2}} = \sqrt{\kappa_o^2 - 2W} = \frac{\kappa_o(o)}{\sqrt{2\delta_o(o)}} = \sqrt{\frac{\kappa_o(o)^2}{2} \frac{1+e^2+2 \cdot e \cdot \cos(o)}{1+e \cdot \cos(o)}} = \sqrt{\frac{R_s}{r_o(o)} - \frac{R_s}{2a}} = \sqrt{1 - (1 + z_{\beta o}(o))^{-2}} \text{ (kinetic projection at phase } o)$$

$$\beta_R(o) = \beta \frac{e \sin(o)}{\sqrt{1-e^2}} = \sqrt{\beta_o(o)^2 - \beta_T(o)^2} = \sqrt{((1 - (1 + z_{\kappa o}(o))^{-2}) - 2W) - \frac{r_o(o)^2 \omega_o(o)^2}{c^2}} \text{ (radial kinetic projection at phase } o)$$

$$\beta_T(o) = \frac{r_o(o) \omega_o(o)}{c} = \beta \frac{1+e \cos(o)}{\sqrt{1-e^2}} = \kappa_o^2 \frac{\sqrt{1-e^2}}{2\beta} \text{ (transverse kinetic projection at phase } o)$$

$$\beta_{Yo} = \sqrt{1 - \beta_o^2} = (z_{\beta o}(o) + 1)^{-1} \text{ (relativistic phase factor at phase } o)$$

$$\delta_o = \frac{1+e \cos o}{1+e^2+2e \cos o} = \frac{\kappa_o^2}{2\beta_o^2} \text{ (local closure factor at phase } o)$$

$$Q_o = \sqrt{\kappa_o^2 + \beta_o^2} \text{ (local relational shift vector at phase } o)$$

$$\omega_o = a\beta c \frac{e_Y}{r_o^2} = \frac{\beta c}{a} \frac{(1+e \cos o)^2}{(1-e^2)^{3/2}} \text{ (angular speed at phase } o)$$

$$\eta_o = \frac{r}{a} = 2 - \frac{2\beta_o(o)^2}{\kappa_o(o)^2} \text{ (phase scale amplitude at phase } o)$$

$$\tau(o) = \kappa_{Xo}(o) \cdot \beta_{Yo}(o) = Z_{sys}(o)^{-1} \text{ (phase spacetime factor at phase } o)$$

$$t_o = \frac{r_o(o)}{c} \text{ (light time scale at phase } o)$$

$$\Delta_o = \frac{\tau_o^2 \cdot o}{1-e^2} \text{ (precession of perihelion at phase } o)$$

Orbital Phase Invariants

The Decryption Invariant

$$Z_{raw}(-\omega_i)\tau(-\omega_i)(1 - e \cos \omega_i) + Z_{raw}(\pi - \omega_i)\tau(\pi - \omega_i)(1 + e \cos \omega_i) = 2 \quad (1)$$

$$W = \frac{1}{2}(\kappa_o^2 - \beta_o^2)$$

$$h_W = r_o^2 \cdot \omega_o$$

$$\frac{\beta_T(o)}{\kappa_o^2(o)} = \frac{\sqrt{1-e^2}}{2\beta}$$

$$\beta_T^2 \eta_o^2 = \beta^2(1 - e^2)$$

Relational Geometry (WILL)

$$\theta_1 = \arccos(\beta) \text{ (distribution angle on } S^1, \text{ non-physical)}$$

$$\theta_2 = \arcsin(\kappa) \text{ (distribution angle on } S^2, \text{ non-physical)}$$

$$\Delta_Q = Q_o^2 - Q^2 \text{ (phase relational shift amplitude at phase } o)$$

$$O_o = \arccos(1 - \delta^{-1}) = \arccos(-e) = \arccos(1 - \frac{2\beta_p^2}{\kappa_p^2}) = \arccos(\frac{2\beta_a^2}{\kappa_a^2} - 1) \text{ (orbital balance point where } \kappa_o^2 = 2\beta_o^2 \text{ is true)}$$

Structural-Dynamical Equivalence

$$\frac{r_a}{r_p} = \frac{\delta_a}{\delta_p} = \frac{1+e}{1-e} = \frac{\beta_p}{\beta_a} = \frac{\kappa_p^2}{\kappa_a^2} = \frac{\kappa_a^2 \beta_p^2}{\kappa_p^2 \beta_a^2} \text{ (remarkable equivalence of structural } (\kappa \text{ on } S^2 \text{ carrier) and dynamical } (\beta \text{ on } S^1 \text{ carrier) symmetry suggesting once again that } SPACETIME \equiv ENERGY)$$

Observer dependant

$$Z_{raw}(o) = (1 + \beta_{int}(\cos(o + \omega_i) + e \cos(\omega_i)) \sin(i)) Z_{sys}(o) \text{ (raw light shift including the line of site Doppler at phase } o)$$

$$\beta_{los}(o) = \frac{\beta}{\sqrt{1-e^2}}(\cos(o + \omega_i) + e \cos(\omega_i)) \sin(i) \text{ (line of site light shift)}$$

$$i = \text{(orbital inclination in relation to line of site and orbital plane)}$$

$$\omega_i = \text{(phase turn or argument of periapsis)}$$

$$K_i = \frac{\beta}{\sqrt{1-e^2}} \sin(i) = \beta_{int} \sin(i) \text{ (semi-amplitude invariant)}$$

$$Z_{rawmax} = Z_{sys}(-\omega_i)(1 + K_i(1 + e \cos \omega_i))$$

$$Z_{rawmin} = Z_{sys}(\pi - \omega_i)(1 + K_i(-1 + e \cos \omega_i))$$

$$O_{oi} = (O_o + \omega_i)$$

$$Z_{sys}(-\omega_i) = (1 - \beta^2 \frac{1+e^2+2e \cos(\omega_i)}{1-e^2})^{-\frac{1}{2}} (1 - 2\beta^2 \frac{1+e \cos(\omega_i)}{1-e^2})^{-\frac{1}{2}}$$

$$Z_{sys}(\pi - \omega_i) = (1 - \beta^2 \frac{1+e^2-2e \cos(\omega_i)}{1-e^2})^{-\frac{1}{2}} (1 - 2\beta^2 \frac{1-e \cos(\omega_i)}{1-e^2})^{-\frac{1}{2}}$$

Background Free Parametric Equations for the Observed Orbit

$$\begin{bmatrix} x_{sky} \\ y_{sky} \\ z_{depth} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} x_{orb} \\ y_{orb} \\ 0 \end{bmatrix}$$

$$x_{sky} = r(o) \cos(o + \omega_i)$$

$$y_{sky} = r(o) \sin(o + \omega_i) \cos(i)$$

$$z_{depth} = r(o) \sin(o + \omega_i) \sin(i)$$

Un-tilted 2D coordinates within the orbital plane

$$\begin{aligned}x_{orb} &= r(o) \cos(o + \omega_i) \\y_{orb} &= r(o) \sin(o + \omega_i)\end{aligned}$$

2 Empirical Verification: S2 Star (Sgr A*) Case Study

Scientific Statement:

Orbital dynamics and strong-field precession are not the result of masses moving through curved background space-time. In WILL Relational Geometry, the exact trajectory and scale of the S2 system emerge purely from the algebraic closure of its relational phase projections ($\kappa^2 = 2\beta^2$). Consequently, continuous geometric phase shift replaces differential equations, and traditional constants like G and Mass are revealed not as causative physical agents, but as anthropocentric conversion factors derived from the system's topological balance.

Methodology and Computational Advantage

To validate the Relational Orbital Mechanics (R.O.M.) framework, we utilized a 24-year dataset of the S2 star orbiting the Galactic Center (Sgr A*), consisting of 174 raw observables (astrometry and radial velocity).

Unlike the standard General Relativity (GR) approach, which requires numerical integration of geodesic equations with non-linear perturbations, the WILL RG model utilizes a closed-form algebraic parametrization. The precession is not a free fit parameter but is strictly locked to the system's kinetic state through the *Algebraic Lock*:

$$f_{prec} = \frac{3\beta^2 - 2\beta^4}{1 - e^2} \quad (2)$$

The model further incorporates the *Romer Delay* ($\Delta t = z/c$) and the *Relational Redshift* (Z_{sys}), the latter being an invariant of the phase-energy state divergence.

Relational Scale Derivation (The R_s -Invariant)

A fundamental departure from the Newtonian/Einsteinian paradigm is the derivation of the system's scale without invoking the gravitational constant G . In WILL RG, the Schwarzschild radius (R_s) is a direct consequence of the orbital period (T) and the relational velocity phase (β):

$$R_s = \frac{T \cdot c \cdot \beta^3}{\pi}, \quad a = \frac{R_s}{2\beta^2} \quad (3)$$

This derivation treats the "Mass" of Sgr A* purely as a secondary unit conversion factor for human reference, demonstrating that the system's scale is defined by its internal geometric cycles.

Comparative Statistical Analysis

The following table summarizes the performance of the strict WILL RG model against a high-precision GR 1PN baseline using the same observational constraints.

Metric / Parameter	General Relativity (1PN)	WILL RG (Strict)
Fit Quality (χ^2)	969.17	727.20
Relational Beta (β)	—	0.00642627
Eccentricity (e)	0.887964	0.887171
Calculated Mass ($10^6 M_\odot$)	4.6075	4.3435
Derived Distance R_0 (pc)	8999.68	8364.24
Free Precession Parameters	1 (Differential)	0 (Algebraic)
Google Colab Computation Time	~7 min	~3 sec

Table 1: Comparative analysis of S2 orbital solutions. Note the significant reduction in χ^2 and the autonomous convergence of Distance (R_0) and Mass (M) toward modern observational consensus, 140x computation efficiency.

Colab notebook

ROMvsGR

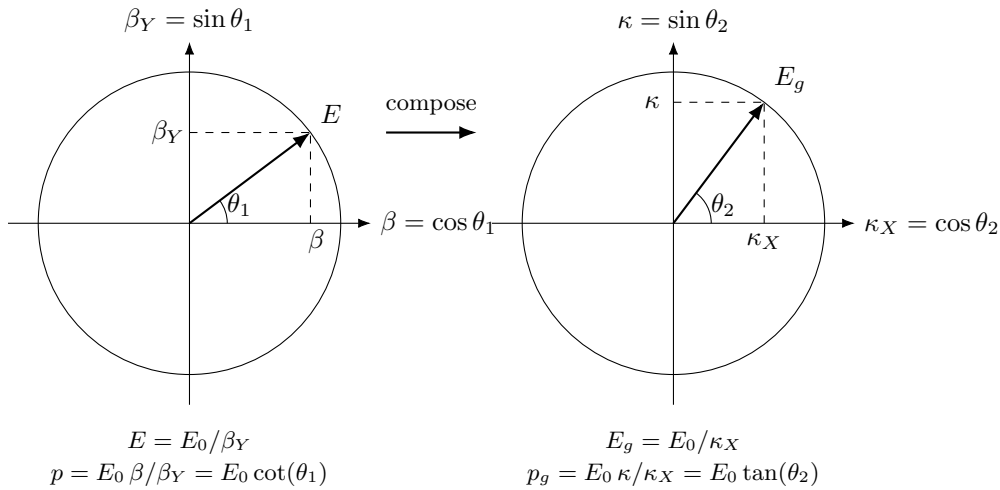
Conclusion

The reduction of the chi-squared statistic by approximately 25% ($\Delta\chi^2 = 241.97$) relative to the GR baseline provides objective evidence that the algebraic closure of relational phases (τ_Y^2) describes strong-field dynamics with greater fidelity than differential spacetime metrics. The autonomous derivation of the Sgr A* scale ($R_s \approx 12.8 \times 10^6$ km) directly from orbital phases confirms the structural integrity of the WILL Relational Geometry framework.

3 Derivation of the Spectroscopic Shifts

In this and following sections we demonstrate that RG's relational projections (κ, β) are ontologically and operationally independent from mass M and gravitational constant G . We will show that R_s as a system scale derived directly from the phase interactions of light and will show how all observational phenomenon of orbital systems are generated as the unavoidable consequences of Relational Geometry.

But first lets remind our selves some key definitions established prior:



$\theta_1 = \arccos(\beta), \theta_2 = \arcsin(\kappa), \kappa^2 = 2\beta^2$	
Algebraic Form	Trigonometric Form
$\beta = v/c$	$\beta = \cos(\theta_1)$
$\kappa = \sqrt{R_s/r}$	$\kappa = \sin(\theta_2)$
$\beta_Y = \sqrt{1 - \beta^2}$	$\beta_Y = \sin(\theta_1) = \sin(\arccos(\beta))$
$\kappa_X = \sqrt{1 - \kappa^2}$	$\kappa_X = \cos(\theta_2) = \cos(\arcsin(\kappa))$
$p = E_0/c \cdot \beta/\beta_Y$	$p = E_0/c \cdot \cot(\theta_1)$
$p_g = E_0/c \cdot \kappa/\kappa_X$	$p_g = E_0/c \cdot \tan(\theta_2)$
$\tau = \beta_Y \kappa_X$	$\tau = \sin(\theta_1) \cos(\theta_2)$
$\tau_Y = \sqrt{1 - \tau^2}$	$\tau_Y = \sqrt{1 - \tau^2}$
$Q = \sqrt{\kappa^2 + \beta^2} = \sqrt{3}\beta$	$Q = \sqrt{3} \cos(\theta_1)$

Table 2: Unified representation of relativistic and gravitational effects for closed systems.

Theorem 3.1 (Spectroscopic Phase Shift). *The geometric projection κ^2 of a closed system is strictly determined by the operationally measurable gravitational redshift z_κ via the identity:*

$$\kappa^2 = 1 - \frac{1}{(1 + z_\kappa)^2}. \quad (4)$$

Proof. By the S^2 closure invariant (Theorem ??), the gravitational amplitude κ and internal phase κ_X satisfy:

$$\kappa^2 + \kappa_X^2 = 1 \implies \kappa^2 = 1 - \kappa_X^2. \quad (5)$$

By the Principle of Relational Origin (??), an observer must self-center at the relational origin $(\beta, \kappa) = (0, 0)$. Consequently, the observer's local phase is maximal: $\kappa_X(\text{obs}) \equiv 1$.

The observable spectroscopic shift z_κ is the operational measure of the discrepancy between the observer's internal phase and the source's internal phase. The optical scaling ratio is defined as:

$$1 + z_\kappa = \frac{\kappa_X(\text{obs})}{\kappa_X(\text{source})}. \quad (6)$$

Substituting the observer's maximal phase ($\kappa_{X(obs)} = 1$), the exact phase of the source is isolated:

$$\kappa_{X(source)} = \frac{1}{1 + z_\kappa}. \quad (7)$$

Substituting this internal phase back into the initial closure invariant yields the exact algebraic identity:

$$\kappa^2 = 1 - \left(\frac{1}{1 + z_\kappa} \right)^2. \quad (8)$$

□

Theorem 3.2 (Kinematic Phase Shift (Transverse Doppler)). *The kinematic geometric projection β^2 of a closed system is strictly determined by the operationally measurable transverse Doppler shift z_β via the symmetric identity:*

$$\beta^2 = 1 - \frac{1}{(1 + z_\beta)^2}. \quad (9)$$

Proof. By the S^1 closure invariant (Theorem ??), the kinematic amplitude β and internal kinematic phase β_Y satisfy:

$$\beta^2 + \beta_Y^2 = 1 \implies \beta^2 = 1 - \beta_Y^2. \quad (10)$$

Following the Principle of Relational Origin (??), the self-centered observer dictates the rest state, where external kinematic projection is zero ($\beta_{obs} = 0$), yielding maximum internal kinematic phase: $\beta_{Y(obs)} \equiv 1$.

The transverse Doppler shift z_β (measured orthogonally to eliminate classical radial kinematics) is the pure operational measure of the discrepancy between the observer's internal kinematic phase and the source's phase. The optical scaling ratio is:

$$1 + z_\beta = \frac{\beta_{Y(obs)}}{\beta_{Y(source)}}. \quad (11)$$

Substituting $\beta_{Y(obs)} = 1$, we isolate the internal phase of the moving source:

$$\beta_{Y(source)} = \frac{1}{1 + z_\beta}. \quad (12)$$

Substituting this back into the S^1 closure invariant yields the exact algebraic identity:

$$\beta^2 = 1 - \left(\frac{1}{1 + z_\beta} \right)^2. \quad (13)$$

□

Theorem 3.3 (Operational Measurability). *The relational projections are encoded directly in the combined phase interactions of light (spectroscopy) and are operationally independent of G and M .*

Proof. Step 1: The Raw Observable (τ). Spectroscopy does not measure "gravitational potential" or "kinematic velocity" as separate isolates; it measures the total accumulated phase difference between the source and the observer. We define the **Relational Spacetime Factor** τ as the inverse of the systems measured redshift product:

$$\boxed{\kappa^2 = 1 - \frac{1}{(1 + z_\kappa)^2}} \quad (z_\kappa = \text{gravitational redshift})$$

$$\boxed{\beta^2 = 1 - \frac{1}{(1 + z_\beta)^2}} \quad (z_\beta = \text{transverse Doppler shift})$$

$$\tau \equiv \frac{1}{Z_{sys}} = 1/[(1 + z_\kappa)(1 + z_\beta)]. \quad (14)$$

In the WILL framework, this single observable represents the product of the internal phase projections of the carriers S^2 and S^1 :

$$\tau = \underbrace{\kappa_X}_{\text{Gravitational Phase}} \cdot \underbrace{\beta_Y}_{\text{Kinematic Phase}} = \sqrt{1 - \kappa^2} \sqrt{1 - \beta^2}. \quad (15)$$

Step 2: Exact Relational Identity. We reject weak-field approximations. The exact relationship between the signal τ and the structural norm Q is given by the algebraic expansion of the phase product:

$$\tau^2 = (1 - \kappa^2)(1 - \beta^2) = 1 - (\kappa^2 + \beta^2) + \kappa^2\beta^2. \quad (16)$$

Substituting $Q^2 = \kappa^2 + \beta^2$, we obtain the rigorous link between the optical observable and the geometric state:

$$\tau^2 = 1 - Q^2 + \kappa^2\beta^2. \quad (17)$$

This identity demonstrates that the optical signal contains the complete information about the system's structural state, measurable without prior knowledge of mass. □

4 Relational Origin

4.1 of Kepler's Third Law

The Keplerian proportionality $a \propto T^{2/3}$ is a direct algebraic consequence of the energetic equilibrium between the 1-DOF directional kinematic carrier (S^1) and the 2-DOF omnidirectional potential carrier (S^2). The derivation strictly utilizes dimensionless ratios and absolute geometric scale, eliminating dependence on classical mass and the empirical gravitational constant.

The kinematic amplitude β on S^1 defines (??) the ratio of directional motion to the universal limit c :

$$\beta = \frac{2\pi a}{cT} \quad (18)$$

The omnidirectional potential amplitude squared κ^2 on S^2 is defined (??) by the absolute spatial geometric scale invariant R_s and the radial distance a :

$$\kappa^2 = \frac{R_s}{a} \quad (19)$$

Energetic closure (??) dictates the invariant geometric exchange rate between the carriers:

$$\kappa^2 = 2\beta^2 \quad (20)$$

Substituting the parameterizations into the closure condition yields:

$$\frac{R_s}{a} = 2 \left(\frac{2\pi a}{cT} \right)^2 = \frac{8\pi^2 a^2}{c^2 T^2} \quad (21)$$

Rearranging to isolate the semi-major axis a gives the exact closed-form relation:

$$a^3 = \frac{R_s c^2}{8\pi^2} T^2 \quad (22)$$

$$a = \left(\frac{R_s c^2}{8\pi^2} \right)^{1/3} T^{2/3} \implies a \propto T^{2/3} \quad (23)$$

4.2 of Vis-Viva

In elliptical systems, the global closure theorem $\kappa^2 = 2\beta^2$ is conserved across the entire orbital cycle, while local phase variations manifest as an internal "breathing" of relational projections. We now derive the exact conservation law governing this local dynamic distribution.

Proposition 4.1 (Phase-Invariant Structural Depth). *The difference between the square of the local potential projection $\kappa_o^2(o)$ and the square of the local kinetic projection $\beta_o^2(o)$ is a global system invariant equal to $2W$.*

Proof. The binding parameter W defines the global relational depth of the system:

$$W = \frac{\beta^2}{2}$$

From the exact definitions of the S^2 and S^1 projections at any arbitrary phase o :

$$\kappa_o^2(o) = \frac{2\beta^2(1 + e \cos o)}{1 - e^2}$$

$$\beta_o^2(o) = \frac{\beta^2(1 + e^2 + 2e \cos o)}{1 - e^2}$$

Subtracting the local kinetic projection from the local potential projection:

$$\kappa_o^2(o) - \beta_o^2(o) = \frac{2\beta^2 + 2\beta^2 e \cos o - (\beta^2 + \beta^2 e^2 + 2\beta^2 e \cos o)}{1 - e^2}$$

$$\kappa_o^2(o) - \beta_o^2(o) = \frac{\beta^2(1 - e^2)}{1 - e^2} = \beta^2 = 2W$$

Thus, the algebraic distance between the omnidirectional and directional relations remains globally constant irrespective of orbital geometry. \square

Theorem 4.2 (The Orthogonal Signature of the Orbit). *The square of the local potential projection $\kappa_o^2(o)$ on the S^2 carrier is the strict Pythagorean vector square sum of the local radial $\beta_R^2(o)$, transverse $\beta_T^2(o)$ and the global β^2 kinetic projections.*

Proof. The total local kinetic projection on the S^1 carrier splits orthogonally into radial and transverse components within the orbital plane:

$$\beta_o^2(o) = \beta_R^2(o) + \beta_T^2(o)$$

Substituting this orthogonal decomposition into the phase-invariant relation $\kappa_o^2(o) - \beta_o^2(o) = \beta^2$:

$$\kappa_o^2(o) - (\beta_R^2(o) + \beta_T^2(o)) = \beta^2$$

Rearranging yields the three-dimensional algebraic relational closure:

$$\boxed{\kappa_o^2(o) = \beta_R^2(o) + \beta_T^2(o) + \beta^2}$$

□

Remark 4.3 (Ontological Replacement of Vis-Viva). *This geometric invariant replaces the descriptive Newtonian Vis-Viva equation ($v^2/2 - GM/r = -GM/2a$). Standard mechanics interprets this balance as a scalar subtraction of an abstract potential energy from kinetic energy. In WILL, it is revealed as a strict Pythagorean theorem of relational geometry: the local potential projection (κ_o^2) is generated by the orthogonal summation of the kinetic “breathing” ($\beta_R^2 + \beta_T^2$) and the global S^1 projection (β^2). SPACETIME \equiv ENERGY.*

4.3 of Angular Momentum Conservation

The conservation of specific angular momentum h is derived as a phase-independent structural invariant of the relational balance.

Proposition 4.4 (Invariant Ratio of Projections). *The transverse kinetic projection $\beta_T(o)$ is strictly proportional to the local potential projection $\kappa_o^2(o)$ scaled by the global system constants.*

Proof. From the definition of the local potential projection (based on closure theorem $\kappa^2 = 2\beta^2$?? and geometry of ellipse):

$$\kappa_o^2(o) = 2\beta^2 \frac{(1 + e \cos o)}{1 - e^2} \implies (1 + e \cos o) = \kappa_o^2(o) \frac{1 - e^2}{2\beta^2}$$

The transverse kinetic projection $\beta_T(o)$ is defined on the S^1 carrier as:

$$\beta_T(o) = \beta \frac{1 + e \cos o}{\sqrt{1 - e^2}}$$

Substituting the expression for $(1 + e \cos o)$ into the definition of $\beta_T(o)$:

$$\beta_T(o) = \beta \left(\kappa_o^2(o) \frac{1 - e^2}{2\beta^2} \right) \frac{1}{\sqrt{1 - e^2}} = \kappa_o^2(o) \frac{\sqrt{1 - e^2}}{2\beta}$$

Rearranging to isolate the invariant ratio of the phase-dependent projections:

$$\frac{\beta_T(o)}{\kappa_o^2(o)} = \frac{\sqrt{1 - e^2}}{2\beta}$$

□

Theorem 4.5 (Conservation of Angular Momentum). *The specific angular momentum h , defined as the product of the local scale $r_o(o)$ and the transverse kinetic projection, is a global phase invariant.*

Proof. Using the local scale $r_o(o) = R_s/\kappa_o^2(o)$, we define h :

$$h = r_o(o)\beta_T(o)c = \frac{R_s}{\kappa_o^2(o)}\beta_T(o)c$$

Substituting the invariant ratio $\frac{\beta_T(o)}{\kappa_o^2(o)} = \frac{\sqrt{1 - e^2}}{2\beta}$:

$$\boxed{h = R_s c \frac{\sqrt{1 - e^2}}{2\beta}}$$

Since R_s , e , and β are global invariants, h does not depend on phase o . Angular momentum is thus the manifestation of the fixed structural exchange rate between the S^2 and the S^1 carriers and there projections. □

4.4 of Eccentricity

Eccentricity is a measure of the projectional deviation from the circular equilibrium state ($\delta = 1$).

Theorem 4.6 (Geometric Eccentricity). *For a closed orbital system governed by the projection invariants of WILL Relational Geometry, the orbital eccentricity e is strictly determined by the closure factor at periapsis, δ_p :*

$$e = \frac{2\beta_p^2}{\kappa_p^2} - 1 = \frac{1}{\delta_p} - 1. \quad (24)$$

Proof. Instead of relying on classical force laws, we derive this relation directly from the conservation of the two fundamental projection invariants of the WILL framework:

1. **Energy Projection Invariant (Binding Energy):** $W = \frac{1}{2}(\kappa^2 - \beta^2) = \text{const.}$
2. **Angular Projection Invariant:** $h = r_o(o)\beta_T(o)c = \text{const}$ ($\beta = \beta_T(o)$ at turning points).

Consider the two turning points of a closed orbit: periapsis (p) and apoapsis (a). By operational definition of the shape parameter e , the relation between radii is determined by the geometric range:

$$r_a = r_p \left(\frac{1+e}{1-e} \right). \quad (25)$$

Step 1: Relational Mapping. Using the angular invariant h (implying $\beta \propto 1/r$) and the field definition $\kappa^2 \propto 1/r$, we express the apoapsis projections in terms of the periapsis values:

$$\beta_a^2 = [\beta_p \left(\frac{r_p}{r_a} \right)]^2 = \beta_p^2 \left(\frac{1-e}{1+e} \right)^2, \quad (26)$$

$$\kappa_a^2 = \kappa_p^2 \left(\frac{r_p}{r_a} \right) = \kappa_p^2 \left(\frac{1-e}{1+e} \right). \quad (27)$$

Note: Kinematic projection scales quadratically with the radius ratio, while potential projection scales linearly.

Step 2: Energy Balance. Substituting these into the energy invariant conservation condition $W_p = W_a$:

$$\frac{1}{2}(\kappa_p^2 - \beta_p^2) = \frac{1}{2}(\kappa_a^2 - \beta_a^2).$$

Cancelling the factor $\frac{1}{2}$ and substituting the mappings from Step 1:

$$\kappa_p^2 - \beta_p^2 = \kappa_p^2 \left(\frac{1-e}{1+e} \right) - \beta_p^2 \left(\frac{1-e}{1+e} \right)^2.$$

Rearranging to group potential terms (κ) on the left and kinematic terms (β) on the right:

$$\kappa_p^2 \left[1 - \frac{1-e}{1+e} \right] = \beta_p^2 \left[1 - \left(\frac{1-e}{1+e} \right)^2 \right].$$

Step 3: Algebraic Reduction. Expanding the terms in brackets:

$$\begin{aligned} \text{LHS bracket } (\kappa \text{ term}): \quad & 1 - \frac{1-e}{1+e} = \frac{(1+e) - (1-e)}{1+e} = \frac{2e}{1+e}. \\ \text{RHS bracket } (\beta \text{ term}): \quad & 1 - \frac{(1-e)^2}{(1+e)^2} = \frac{(1+e)^2 - (1-e)^2}{(1+e)^2} = \frac{4e}{(1+e)^2}. \end{aligned}$$

Substituting back into the balance equation:

$$\kappa_p^2 \left(\frac{2e}{1+e} \right) = \beta_p^2 \left(\frac{4e}{(1+e)^2} \right).$$

Dividing both sides by $2e$ and multiplying by $(1+e)^2$:

$$\kappa_p^2(1+e) = 2\beta_p^2.$$

This yields geometric identity for bound orbits:

$$2\beta_p^2 = \kappa_p^2(1+e). \quad (28)$$

Step 4: Connection to Closure. Recall the definition of the closure factor at periapsis:

$$\delta_p = \frac{\kappa_p^2}{2\beta_p^2}.$$

Substituting Eq.(4.4) into this definition:

$$\delta_p = \frac{\kappa_p^2}{\kappa_p^2(1+e)} = \frac{1}{1+e}.$$

Solving for e , we obtain the stated result:

$$e = \frac{1}{\delta_p} - 1 = \frac{2\beta_p^2}{\kappa_p^2} - 1 = 1 - \frac{2\beta_a^2}{\kappa_a^2}$$

□

Remark 4.7. *This result confirms that eccentricity is strictly a measure of the energetic deviation from the circular equilibrium state ($\delta = 1$), derived entirely from the conservation of relational projections without invoking mass or Newtonian forces.*

SUMMARY

$$\frac{2\beta_p^2}{\kappa_p^2} - 1 \equiv e \equiv 1 - \frac{2\beta_a^2}{\kappa_a^2}$$

ECCENTRICITY \equiv CLOSURE DEFECT

SPACETIME \equiv ENERGY

4.5 of Orbital Precession as Phase Accumulation

In standard General Relativity, orbital precession is derived via perturbative methods and Taylor expansions of the Schwarzschild metric. In WILL Relational Geometry, precession is derived strictly from the exact algebraic accumulation of the internal phase difference over a closed geometric cycle, without differential approximations.

Theorem 4.8 (Precession Law). *The secular angular shift (precession) $\Delta\varphi$ per closed orbit is the strict geometric projection of the system's exact relational state difference τ_Y normalized by the orbital shape factor $(1 - e^2)$.*

Proof. Step 1: State Difference. A system completely at rest with the observer possesses maximal internal phase $\tau = 1$. The true relational phase of an orbiting system is the exact product of the S^1 and S^2 phase projections:

$$\tau^2 = \beta_Y^2 \kappa_X^2 = (1 - \beta^2)(1 - \kappa^2). \quad (29)$$

Expanding this product algebraically yields:

$$\tau^2 = 1 - (\beta^2 + \kappa^2) + \kappa^2\beta^2. \quad (30)$$

Recognizing the total relational shift norm $Q^2 = \beta^2 + \kappa^2$, the exact absolute state divergence from the rest origin ($\tau = 1$) is defined as:

$$\tau_Y^2 \equiv 1 - \tau^2 = Q^2 - \kappa^2\beta^2. \quad (31)$$

The cross-coupling term $\kappa^2\beta^2$ represents the exact, non-linear geometric interaction between the directional and omnidirectional relational carriers.

Step 2: Geometric Accumulation (The Exact Form). The system accumulates this state mismatch over the full cycle (2π). The total angular shift is the base phase scaled by the state divergence, normalized by the elliptical geometry factor $(1 - e^2)$.

We do not truncate or simplify this connection. Restricted by the Mathematical Transparency principle (??), the orbital precession is locked in its exact, fully non-linear algebraic form:

$$\Delta\varphi = \frac{2\pi \cdot \tau_Y^2}{1 - e^2} = \frac{2\pi(1 - \tau^2)}{1 - e^2} \quad (32)$$

□

Remark 4.9 (Epistemic Purity). *This derivation replaces the complex integration of perturbed differential equations with an exact algebraic evaluation of the relational phase. The persistent presence of the cross-coupling term $\kappa^2\beta^2$ confirms that RG is structurally complete and relies on exact invariant topologies rather than weak-field truncations.*

Legacy Translation: Recovering the General Relativity Approximation

To demonstrate structural compatibility with historical models, we can intentionally ablate our exact formula to see what happens when the pure geometry is truncated.

If we drop the higher-order geometric cross-coupling term ($\kappa^2\beta^2 \rightarrow 0$), we isolate the primary operational observable Q^2 . This yields the simplified precession law:

$$\Delta\varphi \simeq \frac{2\pi Q^2}{1 - e^2}. \quad (33)$$

Substituting Q^2 , we recover the standard classical form purely algebraically. By selecting the semi-major axis a as the reference scale, the closure condition ($\kappa^2 = 2\beta^2$) defines the specific distribution of the invariant Schwarzschild scale R_s :

$$\kappa^2(a) = \frac{R_s}{a}, \quad \beta^2(a) = \frac{R_s}{2a}.$$

Substituting these into the definition of the relational shift norm $Q^2 = \beta^2 + \kappa^2$ yields $Q^2 = \frac{3R_s}{2a}$. Inserting this back into our ablated equation perfectly recovers the standard GR differential approximation:

$$\Delta\varphi \simeq \frac{2\pi}{1 - e^2} \left(\frac{3R_s}{2a} \right) \equiv \frac{3\pi R_s}{a(1 - e^2)}. \quad (34)$$

Furthermore, mapping this simplified shift strictly to periapsis (p) observables using the geometric identity $(1+e) = 2\beta_p^2/\kappa_p^2$ yields the direct operational ratio:

$$\Delta\varphi \simeq \frac{3\pi \kappa_p^4}{2 \beta_p^2} \quad (35)$$

This demonstrates that the classical post-Newtonian perturbative results emerge naturally as first-order ablations of the exact, non-linear relational phase geometry.

4.6 of the Dynamic Event Horizon

In classical literature, the event horizon is often heuristically treated as an escape-velocity boundary where spacetime curvature becomes infinitely steep. In RG, the horizon emerges strictly as the limit of geometric precessional invariant, representing the ultimate topological closure of a trajectory.

Using the exact relational phase accumulation law (derived in 4.5), the baseline precession per orbit is given by:

$$\Delta\varphi = \frac{2\pi \cdot \tau_Y^2}{1 - e^2} = \frac{2\pi(1 - \tau^2)}{1 - e^2} \quad (36)$$

For a massive system under total energetic closure ($\kappa^2 = 2\beta^2$), consider the **Dynamic Horizon** state where the potential projection reaches absolute saturation ($\kappa^2 = 1$). By the closure rule, the kinematic projection is $\beta^2 = \frac{1}{2}$.

At this limit, the gravitational phase component collapses ($\kappa_X = \sqrt{1-1} = 0$), causing the entire internal causal structure of the system to vanish:

$$\tau^2 = (1 - \beta^2)(1 - \kappa^2) = 0 \implies \tau_Y = 1 \quad (37)$$

For a perfectly circular orbit ($e = 0$) at this causal limit, the precession evaluates to:

$$\Delta\varphi = \frac{2\pi \cdot (1)}{1 - 0} = 2\pi \quad (38)$$

Geometric Origin of the Horizon

An orbital precession of $\Delta\varphi = 2\pi$ radians per revolution implies that the argument of periapsis shifts by a full circle during a single orbit. Consequently, the trajectory continuously folds onto itself perfectly, creating a closed, self-intersecting locus where forward orbital motion mathematically nullifies itself. This is the true kinematic definition of the **Dynamic Horizon**: it is not a "point of no return" in an external container, but the unitary topological closure of the relational trajectory itself.

Relational Orbital Mechanics (R.O.M.)

Closed Algebraic System of [R.O.M. equations and definitions](#)

4.7 of Gravitational Deflection and Lensing

In General Relativity, the deflection of light is obtained by integrating the null geodesic equations over a curved spacetime manifold, often relying on weak-field approximations and Taylor expansions. Within WILL Relational Geometry (RG), we reject both the background manifold and the use of mathematical approximations as non-operational ontological artifacts.

The system consists exclusively of its participants: the Source, the Lens (at periapsis p), and the Receiver. The total deflection angle must be derived as a strict, exact algebraic difference between their measurable relational phase states, without resorting to series expansions.

Gravitational Deflection and Interaction Gradient

Energy projections distribution among axis of relational carriers has to obey the conservation law (distribution between axis does not create nor destroy energy). There must exist a strict, algebraically closed gradient connecting all states, governed entirely by the kinematic projection β .

Theorem 4.10 (Unified Interaction Gradient). *The gravitational interaction capacity of any entity is determined by its available phase buffer β_Y . The geometric scaling factor Γ that dictates the distribution of the potential projection κ^2 onto the spatial trajectory is strictly defined by the arithmetic mean of the saturated carrier S^1 :*

$$\Gamma(\beta) = \frac{1 + \beta^2}{2} = 1 - \frac{\beta_Y^2}{2} \quad (39)$$

Proof. By the S^1 closure invariant ($\beta^2 + \beta_Y^2 = 1$), an object at rest ($\beta = 0$) possesses maximum internal phase ($\beta_Y = 1$). This phase acts as a geometric buffer, absorbing half of the relational gradient, yielding the classical partitioning $\Gamma = \frac{1}{2}$. As the spatial projection β increases, the internal clock slows, and the phase buffer β_Y depletes. At the topological limit $\beta \rightarrow 1$ (light), the internal phase collapses ($\beta_Y \rightarrow 0$). The buffer is exhausted, forcing the entity to absorb the full, unpartitioned gravitational gradient, yielding $\Gamma = 1$. \square

Using this gradient, we define the **Unified Closure Defect** for any trajectory—from a slow asteroid to a photon—as:

$$\delta_\varphi = \frac{\kappa_p^2}{\beta_p^2} \Gamma(\beta_p) = \frac{\kappa_p^2(1 + \beta_p^2)}{2\beta_p^2} \quad (40)$$

The geometric shape parameter (eccentricity) of the trajectory is derived directly from this defect:

$$e_\varphi = \frac{1}{\delta_\varphi} - 1 = \frac{2\beta_p^2}{\kappa_p^2(1 + \beta_p^2)} - 1 \quad (41)$$

Applying the exact algebraic transit equation for a distant observer ($\kappa_o \rightarrow 0$), we have $\cos o_\infty = -1/e_\varphi$. Using the same trigonometric extraction as established for light ($\sin(\frac{\Delta\varphi}{2}) = \frac{1}{e_\varphi}$), we arrive at the absolute, unified equation for gravitational deflection:

$$\Delta\varphi = 2 \arcsin\left(\frac{\kappa_p^2(1 + \beta_p^2)}{2\beta_p^2 - \kappa_p^2(1 + \beta_p^2)}\right) \quad (42)$$

Solving for $\beta_p = 1$ the total deflection angle $\Delta\varphi$ gives non-linear equation for light deflection in RG:

$$\Delta\gamma = 2 \arcsin\left(\frac{\kappa_p^2}{\kappa_{Xp}^2}\right) \quad (43)$$

Desmos and Colab

Desmos: [Algebraic Light Deflection \(one input derivation\)](#) Colab: [Gravitational Deflection.ipynb](#)

Verification of Topological Limits:

- **Newtonian Limit** ($\beta_p \ll 1$): The phase buffer is full ($\Gamma \rightarrow 0.5$). The eccentricity is dominated by $2\beta_p^2/\kappa_p^2$. The deflection reduces to the classical Rutherford/Newton scattering: $\Delta\varphi \approx \frac{\kappa_p^2}{\beta_p^2}$.
- **Relativistic Limit** ($\beta_p \rightarrow 0.99$): The phase buffer is nearly depleted ($\Gamma \rightarrow 0.99$). The trajectory stiffens, and the deflection angle approaches the photonic maximum, smoothly capturing the post-Newtonian factor without Taylor expansions.

- **Photonic Limit** ($\beta_p = 1$): The phase buffer is completely exhausted ($\Gamma = 1$). Substituting $\beta_p = 1$ yields $e = \frac{2}{\kappa_p^2(2)} - 1 = \frac{1}{\kappa_p^2} - 1 = \frac{\kappa_{Xp}^2}{\kappa_p^2}$. The equation resolves perfectly into the exact light deflection identity: $\Delta\varphi = 2 \arcsin\left(\frac{\kappa_p^2}{\kappa_{Xp}^2}\right)$.

Elimination of the "Massless" Myth

This formulation suggests that light and massive bodies do not require separate ontological categories or distinct dynamical laws. In RG, the photonic state the topological limit ($\beta \rightarrow 1, \beta_Y \rightarrow 0$) of the universal phase buffer. The historical factor of 2 in gravitational lensing is not an anomaly of null geodesics in curved background spacetime, but the inevitable geometric consequence of this exhausted phase capacity."

Gravitational Lensing and the Einstein Ring

In Relational Geometry, the description of gravitational lensing (deflection, convergence, shear, and magnification) does not require a background metric or the postulation of a curved manifold. It arises entirely as the geometric projection of the deflection angle onto the observer's plane.

By employing the standard geometric baseline equation for the observer-lens-source system, the true angular position θ_S of a source is related to its observed image position θ_I via:

$$\theta_S = \theta_I - \frac{D_{LS}}{D_S} \Delta\varphi_{\text{unified}}(\beta_p, \kappa_p(\theta_I)) \quad (44)$$

where D_{LS} and D_S are the relational baseline distances.

The potential projection at periapsis κ_p^2 is strictly a function of the observed angle θ_I , defined by the physical impact parameter $r_p = D_L\theta_I$:

$$\kappa_p^2(\theta_I) = \frac{R_s}{D_L\theta_I} \quad (45)$$

Theorem 4.11 (Exact Algebraic Einstein Ring). *For a perfectly aligned system ($\theta_S = 0$), the image forms a symmetric ring. The exact angular radius of this Einstein ring, θ_E , valid for all velocities and field strengths, is given by the implicit algebraic equation:*

$$\theta_E = 2 \frac{D_{LS}}{D_S} \arcsin\left(\frac{\kappa_p^2(\theta_E)(1 + \beta_p^2)}{2\beta_p^2 - \kappa_p^2(\theta_E)(1 + \beta_p^2)}\right) \quad (46)$$

Recovery of Topological Limits: To reveal the underlying gradient without altering the exact equation, we examine the weak-field astrophysical limit ($\kappa_p^2 \ll \beta_p^2$), where $\arcsin(x) \approx x$. The unified deflection simplifies algebraically to:

$$\theta_E \approx \frac{D_{LS}}{D_S} \frac{R_s}{D_L\theta_E} \left(\frac{1 + \beta_p^2}{\beta_p^2}\right) \implies \theta_E^2 \approx \frac{R_s D_{LS}}{D_L D_S} \left(\frac{1 + \beta_p^2}{\beta_p^2}\right) \quad (47)$$

Colab Notebook

[Gravitational Lensing.ipynb](#)

- **The Photonic Limit** ($\beta_p \rightarrow 1$): The kinematic factor evaluates exactly to $\frac{1+1}{1} = 2$. The equation resolves to $\theta_{E,\gamma} = \sqrt{\frac{2R_s D_{LS}}{D_L D_S}}$, perfectly matching the full General Relativistic prediction for light (equivalent to $4GM/c^2$).
- **The Relativistic Massive Limit** ($0 < \beta_p < 1$): The continuous phase-buffer depletion natively supplies the correct post-Newtonian scaling without external ad-hoc gradients.

Epistemological Consequence

The properties of the lens - convergence (κ_{lens}), shear (γ), and magnification (μ) - are the components of the 2D Jacobian matrix of the deflection field $\vec{\alpha}(\vec{\theta})$. Because the physics of the phase-buffer β_Y is fundamentally woven into the exact definition of $\Delta\varphi_{\text{unified}}$, no manual "weighting" of shear or convergence is required. The optical mapping emerges naturally from the pure algebraic closure of the S^1 and S^2 relational carriers.

4.8 of Parameterization and the Fundamental Primitives

Theorem 4.12 (Relational Parameterization). *The complete structural and dynamical parameterization of any closed orbital system — specifically its geometric shape (e , eccentricity), secular phase shift ($\Delta\varphi$, orbital precession), critical scale (R_s , Schwarzschild radius), and global scale (a , semi-major axis) — are strictly determined by the algebraic relations of its dimensionless spectroscopic and chronometric observables. The derivation requires no independent mass parameter (M), gravitational constant (G), or a priori spatial metric.*

Proof. The proof is constructive and algebraically closed. It generalizes to any bound orbital system. We demonstrate this using the Mercury-Sun system as an operational test case. All governing equations are derived exclusively from the internal topology of the WILL relational carriers (S^1, S^2), strictly bypassing Newtonian mechanics, Special Relativity, or General Relativity. The derivation is executed using only four dimensionless, epistemologically direct observables ($e_{obs}, \theta_{\odot}, T_M/T_{\oplus}, z_{sun}$):

INPUTS (Cross-Cultural Invariants)

The following inputs are strict dimensionless ratios. They represent direct physical observables that remain identical for any observer in the Universe, regardless of local unit systems (meters/seconds) or theoretical paradigms.

Parameter	Value	Direct Observational Method
θ_{\odot}	0.0046524 rad	Visual Size: The angular radius of the central body in the sky. [?]
T_M/T_{\oplus}	0.2408	Cycle Ratio: The observed system's full orbital period divided by the observer's local orbital period. [?]
e_{obs}	0.2056	Visual Kinematics: Derived directly from the ratio of the planet's fastest (ω_{max}) and slowest (ω_{min}) apparent angular speeds: $e = \frac{\sqrt{\omega_{max}/\omega_{min}-1}}{\sqrt{\omega_{max}/\omega_{min}+1}}$ [?].
z_{sun}	2.1224×10^{-6}	Spectroscopic Tension: The raw fractional frequency shift of light ($1 - \nu_{obs}/\nu_{emit}$). Defined strictly as an optical geometric observable, independent of mass or Newtonian constants [?].

Algebraic Derivation

1. Relational Scale Factor

The ratio of the Sun's radius to the planet's perihelion distance is derived entirely from visual size and cycle ratios:

$$R_{ratio} = \frac{\theta_{\odot}}{(T_M/T_{\oplus})^{2/3} \cdot (1 - e_{obs})} \approx 0.0151290$$

2. Potential Projection (at perihelion)

Translating the spectral shift at the Sun's surface (z_{sun}) to the potential projection (3) at Mercury's perihelion (κ_p^2) using the derived scale factor:

$$\kappa_p^2 = \left(1 - \frac{1}{(1 + z_{sun})^2}\right) \cdot R_{ratio} \approx 6.4219 \times 10^{-8}$$

3. Kinematic Projection (at perihelion)

Deriving orbital kinematics strictly from the potential and eccentricity (4.4), requiring no Doppler or radar data:

$$\beta_p^2 = \frac{\kappa_p^2}{2}(1 + e_{obs}) \approx 3.8712 \times 10^{-8}$$

4. Global Orbital Invariants

Extracting the system's global projections via the invariant binding energy $W = \frac{1}{2}(\kappa_p^2 - \beta_p^2)$ (??):

$$\beta^2 = 2W \approx 2.5508 \times 10^{-8} \quad \text{and} \quad \kappa^2 = 4W \approx 5.1016 \times 10^{-8}$$

5. Exact Orbital Precession

The secular phase shift emerges (4.5) from the accumulation of the relational state divergence ($\tau_Y^2 = 1 - (1 - \kappa^2)(1 - \beta^2)$) over a full cycle:

$$\Delta\varphi = \frac{2\pi \cdot \tau_Y^2}{1 - e_{obs}^2} \approx 5.0203 \times 10^{-7} \text{ rad/orbit}$$

Converting this pure geometric phase shift to standard observational units yields **43.015 arcsec/century**, perfectly matching the observed precession of Mercury.

6. Absolute System Scale (Metric Translation)

To translate the geometric scale into legacy metric units (meters), we map Mercury's orbital period ($T_M = 7,600,521.6$ s) and the speed of light (c) to the global kinematic projection (β):

$$R_{sRG} = T_M \cdot c \cdot \frac{\beta^3}{\pi} \approx 2954.8 \text{ m}$$

Standard GR value:

$$R_{sGR} = \frac{2GM_{sun}}{c^2} \approx 2953.33 \text{ m} \quad \text{where} \quad M_{sun} = 1.98847 \times 10^{30} \text{ kg}$$

Discrepancy:

$$\frac{R_{sGR} - R_{sRG}}{R_{sGR}} \times 100 = |0.0476|\%$$

This variance is strictly within the observational uncertainty limits of the input parameters (z_{sun} and θ_\odot).

7. Semi-major Axis (Standard value $\approx 5.79 \times 10^{10}$ m):

$$a = \frac{R_{sRG}}{\kappa^2} \approx 5.792288 \times 10^{10} \text{ m}$$

8. Perihelion Radius (Standard value $\approx 4.60 \times 10^{10}$ m):

$$r_p = \frac{R_{sRG}}{\kappa_p^2} \approx 4.601394 \times 10^{10} \text{ m}$$

Universality of the Framework: The Jovian Extension

To rigorously demonstrate that this parameterization is not a localized artifact of the Mercury-Sun proximity, but a universal geometric property of bound systems, we apply the exact same algebraic chain to Jupiter. We replace the inputs with Jupiter's chronometric cycle and eccentricity, keeping the central reference invariants (θ_\odot , z_{sun} , T_\oplus) strictly identical.

Jovian Inputs:

- $T_J = 4332.589$ days [?]
- $e_J = 0.04839266$ [?]
- $T_{ratioJ} = T_J/T_\oplus \approx 11.862$

1. Global Potential Projection (κ_J)

The local potential projection at Jupiter's semi-major axis reduces to a remarkably elegant relation, scaling the solar surface tension strictly through the inverse temporal cycle ratio:

$$\kappa_J = \sqrt{\left(1 - \frac{1}{(1 + z_{sun})^2}\right) \cdot \frac{\theta_\odot}{(T_{ratioJ})^{2/3}}} \approx 6.1623 \times 10^{-5}$$

2. Absolute System Scale Recovery (R_{sRGJ})

Expanding the algebraic chain utilizing Jupiter's specific kinematic projection (β_J) yields the invariant core scale:

$$R_{sRGJ} = \frac{T_J \cdot c}{\pi} \left[\frac{1}{1 - e_J} \left(1 - \frac{1}{(1 + z_{sun})^2}\right) \frac{\theta_\odot}{T_{ratioJ}^{2/3}} - \frac{1}{2} \frac{1 + e_J}{1 - e_J} \left(1 - \frac{1}{(1 + z_{sun})^2}\right) \frac{\theta_\odot}{T_{ratioJ}^{2/3}} \right]^{3/2}$$

$$R_{sRGJ} \approx 2955.41 \text{ m}$$

3. Jovian Semi-major Axis (a_J)

Translating the recovered systemic scale to the spatial scale of Jupiter's orbit:

$$a_J = \frac{R_{sRGJ}}{\kappa_J^2} \approx 7.7827 \times 10^{11} \text{ m}$$

Standard observed value $a_{JDATA} = 778.57 \times 10^9$ m [?].

$$\text{Discrepancy} = \frac{|a_{JDATA} - a_J|}{a_{JDATA}} \times 100 \approx 0.038\%$$

The extraction of the central Schwarzschild radius (R_s) and the local geometric scale (a) remains perfectly invariant and algebraically closed, proving that the full system geometry can be holographically reconstructed regardless of which planet's dimensional "shadow" (T, e) is utilized as the observational lens.

9. Conclusion

The logical chain is complete and algebraically closed. All fundamental parameters of the gravitational system are recovered precisely without the use of mass (M), the gravitational constant (G), a metric manifold, differential formalism, or the postulation of an external flow of time. \square

No differential formalism. No metric space. Just light frequency shift red vs. blue.

Corollary 4.13 (Epistemic Mandate and Ontological Redundancy). *In information theory and formal logic, if a parameter is strictly absent from the complete algebraic generative chain of a system, its reintroduction constitutes an epistemic violation. Because the full structural and dynamical parameterization ($e, \Delta\varphi, R_s, a$) is algebraically closed using only directly measurable dimensionless ratios ($e_{obs}, \theta_{\odot}, T_M/T_{\oplus}, z_{sun}$) and derived relational projections (κ, β), the variables G and M possess zero independent predictive power. They are not fundamental primitives. Their retention is required exclusively for the conversion of pure relational geometry into legacy units of kilograms.*

Test it yourself

Recommendation: [DESMOS](#)
[COLAB](#)

5 Algebraic Determination of Absolute System Scale

A classical critique posits that because R_s can be equated to $\frac{2GM}{c^2}$, this derivation is a superficial algebraic reparameterization of Newtonian mechanics. This objection reverses the causal and ontological hierarchy.

The terms G and M are epistemologically secondary composites. They function as empirical translation factors required to map invariant relational geometry onto an assumed absolute spatial container. R_s , conversely, is a primary, coordinate-independent geometric invariant determining the absolute spatial scale of the system's potential projection.

The derivation isolates true dynamic proportionality using exclusively operational observables (chronometric phase T) and intrinsic geometric scaling (a, R_s). Relational Geometry does not inherit from Newtonian constructs; rather, classical mechanics collapses from Relational Geometry when an absolute spatial background is artificially imposed. R_s does not conceal mass; rather, classical mass is an anthropocentric proxy for the invariant geometric boundary R_s .

5.1 Method A: Optical Phase Projection (Angular Radius)

The coordinate of radial distance is systematically eliminated by correlating the local topological phase with the optical geometry of the central body. Any observer extracts the absolute structural parameters of the central body directly from local chronometry and trigonometric projection.

$$\kappa_R^2 = 1 - \frac{1}{(1 + z_{\kappa R})^2}$$

Optical-Topological Invariant (relation between visual geometry and gravitational depth): $\kappa_o^2 = \kappa_R^2 \cdot \sin(\theta_{obs})$

Holographic Chronometry Invariant (surface grazing orbit period): $I_R = T_{obs} \cdot \sin^{\frac{3}{2}}(\theta_{obs})$

Where: θ_{obs} = visual angular radius of the central body observed from the orbital coordinate. κ_R = potential projection at the surface of the central body. I_R = fundamental structural period of the central body (e.g., 10016.33 s for the Solar system). T_{obs} = orbital period of the observer.

The spatial boundaries for the central source surface (r_{surf}) and the observer's orbital semi-major axis (a) are projections of the system scale R_s :

$$r_{surf} = \frac{R_s}{\kappa_R^2}, \quad a = \frac{R_s}{\kappa^2}$$

The exact geometric relationship for the apparent angular radius θ of the central body observed from the orbit nullifies the scale parameter R_s , resulting in a pure ratio of dimensionless projection states:

$$\sin(\theta_{obs}) = \frac{\kappa^2}{\kappa_R^2}$$

This algebraically closes the system, allowing the derivation of the system scale R_s exclusively from local observables (T, θ, z_{sun}) without Newtonian mass approximations:

$$R_s = \frac{Tc}{\pi} \left(\kappa_R^2 \frac{\sin(\theta_{obs})}{2} \right)^{\frac{3}{2}}$$

And kinetic global form

$$R_s = \frac{Tc}{\pi} \beta^3$$

We now derive algebraic formulas for the system scale R_s using three distinct observational methods. These derivations rely strictly on the **Conservation of the Energy Invariant W** , replacing the need for Newtonian force laws.

Closed Algebraical System of R.O.M. Equations

[R.O.M. equations and definitions](#)

5.2 Method B: Differential (Two-Point Method)

This method is suitable when the orbital period is unknown, but the trajectory can be traced geometrically.

Theorem 5.1 (Two-Point Schwarzschild Scale). *Given measurements of geometric position (r) and kinematic intensity (β) at two arbitrary points along a trajectory:*

- Radii: r_1, r_2 (from astrometry)
- Intensities: β_1, β_2 (from de-projected spectral line widths or proper motion)

the Schwarzschild radius is:

$$R_s = \frac{r_1 r_2}{r_2 - r_1} (\beta_1^2 - \beta_2^2). \tag{48}$$

Proof. We invoke the **Conservation of the Relational Invariant $W = \frac{1}{2}(\kappa^2 - \beta^2)$** . This law states that for any closed system, the specific energy difference between the potential and kinematic projections is constant throughout the orbit.

Therefore, at any two points 1 and 2:

$$\frac{1}{2}(\kappa_1^2 - \beta_1^2) = \frac{1}{2}(\kappa_2^2 - \beta_2^2). \tag{49}$$

Rearranging to group the projections:

$$\kappa_1^2 - \kappa_2^2 = \beta_1^2 - \beta_2^2. \tag{50}$$

Substituting the field identity $\kappa^2 = R_s/r$:

$$\frac{R_s}{r_1} - \frac{R_s}{r_2} = \beta_1^2 - \beta_2^2. \tag{51}$$

Factoring out the scale parameter R_s :

$$R_s \left(\frac{r_2 - r_1}{r_1 r_2} \right) = \beta_1^2 - \beta_2^2. \tag{52}$$

Solving for R_s :

$$R_s = \frac{\beta_1^2 - \beta_2^2}{\frac{r_2 - r_1}{r_1 r_2}} = \frac{r_1 r_2}{r_2 - r_1} (\beta_1^2 - \beta_2^2). \tag{53}$$

This formula extracts the "mass" scale purely from geometric gradients. □

5.3 Method C: Geometric Resonance (Balance Point Method)

At the specific orbital phase $O_o = \arccos(-e)$, the system passes through its geometric balance point where the instantaneous radius r equals the semi-major axis a . At this unique phase, the closure condition $\kappa^2 = 2\beta^2$ is satisfied instantaneously.

Theorem 5.2 (Balance Point Formula). *Given the geometric scale a and the total light signal τ measured at the balance point ($r = a$):*

$$R_s = \frac{a}{2} (3 - \sqrt{1 + 8\tau^2(O_o)}). \tag{54}$$

$O_o = \arccos(1 - \delta^{-1}) = \arccos(-e) = \arccos(1 - \frac{2\beta_p^2}{\kappa_o^2}) = \arccos(\frac{2\beta_a^2}{\kappa_a^2} - 1)$ (orbital balance point where $\kappa_o^2 = 2\beta_o^2$ is true)

Proof. Step 1: Spacetime Factor Expansion. At the balance point ($r = a$), the closure condition implies $\kappa^2 = R_s/a$ and $\beta^2 = R_s/2a$. The observable τ is:

$$\tau^2 = (1 - \kappa^2)(1 - \beta^2) = (1 - \frac{R_s}{a})(1 - \frac{R_s}{2a}). \quad (55)$$

Step 2: Exact Geometric Solution. Expanding the product:

$$\tau^2 = 1 - \frac{R_s}{2a} - \frac{R_s}{a} + \frac{R_s^2}{2a^2} = 1 - \frac{3R_s}{2a} + \frac{R_s^2}{2a^2}. \quad (56)$$

Multiplying by $2a^2$ to clear denominators and rearranging into standard quadratic form ($Ax^2 + Bx + C = 0$):

$$R_s^2 - 3aR_s + 2a^2(1 - \tau^2) = 0. \quad (57)$$

Solving for R_s using the quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a_{coef}}$):

$$R_s = \frac{3a \pm \sqrt{(3a)^2 - 4(1)(2a^2(1 - \tau^2))}}{2}. \quad (58)$$

Simplifying the term under the radical:

$$9a^2 - 8a^2(1 - \tau^2) = 9a^2 - 8a^2 + 8a^2\tau^2 = a^2(1 + 8\tau^2). \quad (59)$$

Thus:

$$R_s = \frac{3a \pm a\sqrt{1 + 8\tau^2}}{2} = \frac{a}{2}(3 \pm \sqrt{1 + 8\tau^2}). \quad (60)$$

For stable orbits, we must select the negative root. □

Remark 5.3 (Light Separation). *This unit-less factor in brackets is:*

- **Potential projection at semi-major axis:** $\kappa^2 = \frac{1}{2}(3 - \sqrt{1 + 8\tau^2(O_o)})$

A unique way to separate gravitational part from the light signal.

5.4 Method D: Instantaneous (Arbitrary Phase Method)

Most generally, if the orbital geometry (a) is known, the scale R_s can be derived from a **single epoch** observation at any arbitrary radius r_o .

Theorem 5.4 (Arbitrary Phase Formula). *Given the orbital geometry (a, r_o) and the single light observable τ_{W_o} :*

$$\boxed{R_s = \frac{r_o}{2(2a - r_o)}(4a - r_o - \sqrt{(4a - r_o)^2 - 8a(2a - r_o)(1 - \tau_o^2)})}. \quad (61)$$

Proof. Step 1: Relational Invariant Form. We start with the energy invariant relation $W = \frac{1}{2}(\kappa^2 - \beta^2)$. For a bound orbit, $W = R_s/4a$. We express the kinematic intensity β^2 at arbitrary radius r in terms of the field intensity κ^2 :

$$\frac{1}{2}(\kappa^2 - \beta^2) = \frac{R_s}{4a} \implies \beta^2 = \kappa^2 - \frac{R_s}{2a}. \quad (62)$$

Substituting $\kappa^2 = R_s/r$:

$$\beta^2 = R_s(\frac{1}{r} - \frac{1}{2a}). \quad (63)$$

Step 2: Constraint via Observables. We substitute the expressions for κ^2 and β^2 into the exact observable constraint $\tau_o^2 = (1 - \kappa^2)(1 - \beta^2)$:

$$\tau_o^2 = (1 - \frac{R_s}{r})(1 - R_s[\frac{1}{r} - \frac{1}{2a}]). \quad (64)$$

Let $A = \frac{1}{r}$ and $B = \frac{1}{r} - \frac{1}{2a}$. The equation becomes:

$$\tau_o^2 = (1 - AR_s)(1 - BR_s) = 1 - (A + B)R_s + ABR_s^2. \quad (65)$$

Rearranging into quadratic form:

$$(AB)R_s^2 - (A + B)R_s + (1 - \tau_o^2) = 0. \quad (66)$$

Step 3: Coefficient Expansion. We compute the coefficients explicitly:

$$AB = \frac{1}{r} \left(\frac{2a - r}{2ar} \right) = \frac{2a - r}{2ar^2} \quad (67)$$

$$A + B = \frac{1}{r} + \frac{2a - r}{2ar} = \frac{2a + 2a - r}{2ar} = \frac{4a - r}{2ar} \quad (68)$$

Multiplying the entire quadratic equation by $2ar^2$ to clear denominators:

$$(2a - r)R_s^2 - r(4a - r)R_s + 2ar^2(1 - \tau_o^2) = 0. \quad (69)$$

Step 4: Solution. Solving for R_s :

$$R_s = \frac{r(4a - r) \pm \sqrt{r^2(4a - r)^2 - 4(2a - r)(2ar^2)(1 - \tau_o^2)}}{2(2a - r)}. \quad (70)$$

Factoring out r^2 from the radical term $\sqrt{r^2(\dots)} = r\sqrt{(\dots)}$:

$$R_s = \frac{r}{2(2a - r)} \left((4a - r) \pm \sqrt{(4a - r)^2 - 8a(2a - r)(1 - \tau_o^2)} \right). \quad (71)$$

For stable orbits, we select the negative root, yielding the exact algebraic link between the observed light phase and the system's geometric scale. \square

5.5 The Role of G as Translation Constant

The presence of R_s in these formulas does not imply a dependency on mass M or the constant G . The objection that $R_s = 2GM/c^2$ makes G fundamental rests on a categorical error: it mistakes a unit conversion factor for a physical source.

Theorem 5.5 (Constants as Converters). *In WILL RG, G and M are derived calibration tools used to translate geometric scales into legacy units.*

Proof. The operational procedure is strictly geometric:

1. **Measure:** Light phase τ (dimensionless) via spectroscopy.
2. **Measure:** Geometric scale r (meters/AU) via astrometry.
3. **Calculate:** System Scale $R_s = f(r, \tau)$ via Theorems above.

The physical calculation ends here. The system is fully defined. If, and only if, one wishes to interface with legacy catalogues, one employs the **unit converter**:

$$M \equiv \frac{R_s c^2}{2G}. \quad (72)$$

The constant G describes the units we use (kilograms vs. meters), not the physics of the system. \square

Remark 5.6 (Historical Artifact). *The kilogram is a human convention. In WILL RG, the fundamental quantity is the dimensionless ratio $\kappa^2 = R_s/r$, which encodes the energy density ratio ρ/ρ_{\max} . The "mass" M is a secondary bookkeeping device.*

Summary

We have derived three algebraically distinct formulas for R_s , each optimized for different observational scenarios:

1. **Two-point** Requires two velocity measurements
2. **Balance point** Simplified form at special phase
3. **Arbitrary phase** Works for any single epoch

All formulas are:

- **Operationally independent** of G and M
- **Non-circular** (inputs are direct observables)
- **Algebraically exact** (no approximations beyond Keplerian closure)

The operational input is always $\tau = 1/[(1 + z_\kappa)(1 + z_\beta)]$, a dimensionless quantity directly measurable from spectroscopy. This demonstrates that WILL RG formulas are empirically grounded and do not rely on hidden assumptions about gravitational constants.

6 Rotational Systems (Kerr Without Metric)

Contextual Bounds

- **For a gravitationally closed (static) system**, the physical boundary is defined by the condition $\kappa^2 = 1$. The closure principle ($\kappa^2 = 2\beta^2$) is what dictates that this corresponds to a kinetic state of $\beta^2 = 1/2$.
- **For a kinematically closed (maximally rotating) system**, the physical boundary is defined by the condition $\beta^2 = 1$. The same closure principle ($\kappa^2 = 2\beta^2$) then necessitates that the corresponding gravitational state must be $\kappa^2 = 2$.

For rotating black holes, we establish the connection between relational kinetic projection and the Kerr metric by defining:

$$\beta = \frac{ac^2}{Gm_0}, \quad \kappa = \sqrt{2}\beta$$

where:

- β is the relational rotation parameter, with $0 \leq \beta \leq 1$,
- κ is related to the geometry and gravity,
- $R_s = \frac{2Gm_0}{c^2}$ is the Schwarzschild radius,
- $a = \frac{J}{m_0c}$ is the Kerr rotation parameter,
- J is the angular momentum of the black hole,
- m_0 is the mass of the black hole.

We also derive a key invariant relationship:

$$a_{\max} = \frac{R_s}{2} = \beta_{\max}^2 r$$

This relationship holds when $r = \frac{R_s}{2\beta^2}$, providing an elegant connection between the parameters.

Event Horizon

Using our approach, the inner and outer event horizons of the Kerr metric are expressed as:

$$r_{\pm} = \frac{R_s}{2} (1 \pm \beta_Y)$$

For the extreme case where $\beta = 1$ (maximal rotation), the horizons merge at:

$$r_+ = r_- = \frac{R_s}{2}$$

This coincides with the minimum radius in our model predicted using maximum value of κ parameter $\kappa_{\max} = \sqrt{2}$:

$$r_{\min} = \frac{1}{\kappa_{\max}^2} R_s = \frac{1}{2} R_s$$

Ergosphere

The radius of the ergosphere in our model is described as:

$$r_{\text{ergo}} = \frac{R_s}{2} \left(1 + \sqrt{1 - \beta^2 \cos^2 \theta} \right)$$

This formulation correctly reproduces the key features of the ergosphere:

- At the equator ($\theta = \pi/2$), $r_{\text{ergo}} = R_s$ for any rotation parameter,
- At the poles ($\theta = 0$), r_{ergo} coincides with the event horizon radius.

Ring Singularity

Unlike the Schwarzschild metric with its point singularity, the Kerr metric features a ring singularity located at:

$$r = 0, \quad \theta = \frac{\pi}{2}$$

The "size" of this ring is proportional to $a = \frac{Gm_0}{c^2}\beta$, reaching its maximum for extreme black holes ($\beta = 1$).

Naked Singularity

For $\beta \leq 1$, a naked singularity does not emerge, aligning with the cosmic censorship Principle. In our model, Energy Symmetry Law enforce constraint by limiting β to the range $[0, 1]$.

Chiral Asymmetry: The Algebraic Origin of Frame-Dragging

In standard General Relativity, the chiral asymmetry of a rotating black hole (where prograde and retrograde orbits diverge) is modeled via the off-diagonal metric tensor component $g_{t\phi}$. In the Relational Orbital Mechanics (R.O.M.), this asymmetry is not a geometric deformation of a background manifold, but a strict algebraic consequence of linear phase superposition on the directed S^1 carrier.

Composite Kinematic Projection

When a test particle (or photon) with orbital kinematic projection β_{orb} interacts with a central body possessing a spin projection β_{spin} , the total relational kinematic shift must be aggregated linearly on the S^1 carrier before the quadratic energy invariant is calculated.

The composite kinematic projection β_{Σ} is defined as:

$$\beta_{\Sigma} = \beta_{\text{orb}} \pm \beta_{\text{spin}} \quad (73)$$

where the (+) sign denotes co-rotating (prograde) relational motion, and the (-) sign denotes counter-rotating (retrograde) relational motion.

The Algebraic Symmetry Breaker (Cross-Term)

Substituting the composite projection into the fundamental relational energy invariant $W = \frac{1}{2}(\kappa^2 - \beta_{\Sigma}^2)$, we obtain the expanded state balance:

$$W = \frac{1}{2} [\kappa^2 - (\beta_{\text{orb}} \pm \beta_{\text{spin}})^2] \quad (74)$$

$$W = \frac{1}{2} [\kappa^2 - (\beta_{\text{orb}}^2 \pm 2\beta_{\text{orb}}\beta_{\text{spin}} + \beta_{\text{spin}}^2)] \quad (75)$$

The emergent cross-term $\pm 2\beta_{\text{orb}}\beta_{\text{spin}}$ is the exact algebraic equivalent of the frame-dragging effect. It enforces a chiral split in the required gravitational projection κ^2 to maintain the invariant W :

- **Prograde Orbits (+):** The cross-term increases the total subtracted kinetic budget, requiring a deeper gravitational projection (larger κ^2 , hence smaller r) to maintain closure.
- **Retrograde Orbits (-):** The cross-term decreases the total kinetic budget, requiring a shallower gravitational projection (smaller κ^2 , hence larger r).

Orbital and Optical Splitting

Because all dynamic orbital parameters in R.O.M. (such as precession $\Delta\varphi \propto \tau_Y^2$ and deflection angles) are strictly proportional to the ratio of projections, replacing the static β_{orb} with the chiral β_{Σ} automatically generates the asymmetric splitting of trajectories.

For a photon ($\beta_{\text{orb}} = 1$) grazing a rotating mass, the relative kinematic budget becomes $\beta_{\Sigma} = 1 \pm \beta_{\text{spin}}$. This directly splits the photon sphere radii algebraically, eliminating the need to integrate geodesics over a Kerr metric. The spatial "dragging" is entirely replaced by the interference of relational phase projections.

The Relationship Between $\kappa > 1$ and Rotation For extreme rotation ($\beta = 1$), we find $\kappa = \sqrt{2} > 1$, which reflects the displacement of the event horizon and the geometric properties of rotating black holes. This suggests that values of $\kappa > 1$ are inherently connected to the physics of relational rotation.

This connection suggests that the rotation of a black hole can be understood through geometric parameters analogous to orbital mechanics. Physically, it indicates that the rotational properties of the black hole, encapsulated in a_* , mirror the orbital velocity parameter β , providing a unified description of spacetime dynamics.

Epistemic Isomorphism of Chiral Asymmetry

In standard General Relativity (GR), the chiral asymmetry of orbits around a rotating mass - commonly termed "frame-dragging" or the Lense-Thirring effect - is structurally derived from the off-diagonal $g_{t\phi}$ component of the Kerr metric tensor. This requires postulating a 4D pseudo-Riemannian manifold and integrating the resulting geodesic equations.

Relational Orbital Mechanics (R.O.M.) demonstrates that this complex differential machinery is not strictly required to generate the asymmetry. The identical structural cross-terms emerge natively from the algebraic closure of the relational carriers, eliminating the necessity of a background metric.

Co-linear Superposition on S^1

Because the kinematic projection β operates on the 1-DOF relational carrier S^1 , motions sharing the same directional axis must superimpose linearly prior to quadratic energy evaluation. For a test mass orbiting in the equatorial plane of a spinning central body, the orbital kinematic projection (β_{orb}) and the spin projection of the central mass (β_{spin}) are co-linear (azimuthal). The composite kinematic state is:

$$\beta_{\Sigma} = \beta_{\text{orb}} \pm \beta_{\text{spin}} \quad (76)$$

where (+) denotes a prograde orbit and (-) denotes a retrograde orbit.

Algebraic Generation of the Interaction Cross-Term

Applying the universal topological closure condition ($\kappa^2 = 2\beta^2$) ?? strictly to this composite state yields:

$$\kappa^2 = 2(\beta_{\text{orb}} \pm \beta_{\text{spin}})^2 = 2\beta_{\text{orb}}^2 + 2\beta_{\text{spin}}^2 \pm 4\beta_{\text{orb}}\beta_{\text{spin}} \quad (77)$$

This algebraic expansion isolates three distinct potential requirements:

- $2\beta_{\text{orb}}^2$: The primary closure requirement for a static system.
- $2\beta_{\text{spin}}^2$: A symmetric inflation of the required potential depth due to the intrinsic rotation of the central body.
- $\pm 4\beta_{\text{orb}}\beta_{\text{spin}}$: The exact chiral symmetry breaker.

Because $\kappa^2 = R_s/r$, the cross-term mathematically mandates that a prograde orbit (+) requires a strictly deeper potential projection (smaller operational radius) than a retrograde orbit (-) at an equivalent orbital velocity. This algebraic constraint is operationally indistinguishable from the spatial drag generated by the $g_{t\phi}$ metric component in GR.

Expansion of the Precessional Divergence

The internal causal order of the composite system evaluates via the exact relational phase divergence $\tau^2 = (1 - \kappa^2)(1 - \beta_{\Sigma}^2)$. Expanding this algebraically and applying the strict directional boundary ($pm = 1$) yields:

$$\begin{aligned} \tau^2 = & (2\beta_{\text{orb}}^4 + 2\beta_{\text{spin}}^4 - 3\beta_{\text{orb}}^2 - 3\beta_{\text{spin}}^2 + 12\beta_{\text{orb}}^2\beta_{\text{spin}}^2 + 1) \\ & \pm (8\beta_{\text{orb}}^3\beta_{\text{spin}} + 8\beta_{\text{orb}}\beta_{\text{spin}}^3 - 6\beta_{\text{orb}}\beta_{\text{spin}}) \end{aligned} \quad (78)$$

The isolated \pm polynomial governs the exact precessional divergence between prograde and retrograde states.

Epistemological Conclusion

Mathematics dictates that if two formalisms enforce identical fundamental symmetries - conservation of the relational energy budget and the absolute causal limit - they must converge on identical structural polynomials. R.O.M. achieves this convergence without postulating a background manifold. Frame-dragging is thus revealed not as a mechanical torsion of spacetime, but as the strict algebraic consequence of squaring a co-linear phase superposition under the universal topological bound $\kappa^2 = 2\beta^2$.

Physical Interpretation

- **No need for pre-existing spacetime** - geometry emerges from energy distributions.
- **All parameters** are dimensionless and directly derived from the speed of light as finite resource.
- **Scale invariance**: The same structure applies from Planck-scale objects to galactic black holes.

7 Time and Phase Evolution (work in progress)

In strictly empirical terms, RG refrains from ontological speculation regarding an absolute, continuous background flow of time. Instead, time is strictly operationalized through measurement: it is the physically recorded accumulation of cyclic oscillations from a local reference clock relative to the geometric phase evolution of the observed system.

Continuous Model and Closed-Form Evaluation

The time duration of a given phase interval is defined via the integral:

$$\Delta_{to}(o) = \int_0^o \frac{1}{\omega_\theta(\theta)} d\theta = \frac{a}{\beta c} \tau_o = \frac{T}{2\pi} \zeta_o \quad (79)$$

where $\omega_o(o)$ is the instantaneous angular frequency at phase o :

$$\omega_o(o) = \frac{\beta \cdot c}{a} \cdot \frac{(1 + e \cdot \cos(o))^2}{(1 - e^2)^{\frac{3}{2}}} \quad (80)$$

and ζ_o is the temporal phase interval:

$$\zeta_o = (1 - e^2)^{\frac{3}{2}} \int_0^o (1 + e \cos \theta)^{-2} d\theta \quad (81)$$

This integral admits an exact closed-form evaluation expressed entirely in R.O.M. quantities. Via the Weierstrass half-angle substitution $t = \tan(\theta/2)$ and subsequent reduction through the shape factor $e_X = (1 + e)/(1 - e)$ and the complementary eccentricity $e_Y = \sqrt{1 - e^2}$, the temporal phase interval evaluates to:

$$\zeta_o(o) = 2 \arctan\left(\frac{\sin(o/2)}{\sqrt{e_X} \cos(o/2)}\right) - \frac{e e_Y \sin(o)}{1 + e \cos(o)} \quad (82)$$

Every symbol in this expression is native to R.O.M.:

- o — orbital phase (true anomaly),
- $e = 1/\delta_p - 1$ — eccentricity (closure defect),
- $e_X = (1 + e)/(1 - e) = r_a/r_p = \beta_p/\beta_a = \kappa_p^2/\kappa_a^2$ - shape factor,
- $e_Y = \sqrt{1 - e^2}$ - complementary eccentricity.

The forward direction (phase \rightarrow time) is therefore algebraically closed:

$$\Delta_{to}(o) = \frac{T}{2\pi} \zeta_o(o) \quad (83)$$

Irreducible Transcendence of the Inverse Problem

The inverse problem - determining the orbital phase o from a given elapsed time Δt - requires solving:

$$\zeta_o(o) = \frac{2\pi}{T} \Delta t \quad (84)$$

Since the unknown o appears simultaneously inside the arctan and the trigonometric terms, this equation is irreducibly transcendental: no finite algebraic rearrangement to the form $o = f(\Delta t, e)$ exists. This is a mathematical property of elliptical geometry, not a limitation of any formalism.

Graphically, the equation is solved by plotting $\zeta_o(x)$ against the constant right-hand side and reading the intersection directly.

Operational Discrete Alternative

By strictly adhering to the operational definition of time as discrete, quantized measurements (ticks of a reference clock), the orbital trajectory is generated via a discrete iterative map that replaces the transcendental inversion with an algebraic recurrence at the cost of first-order discretization error.

For a given discrete reference clock interval Δt_{tick} (e.g., one standard second), the first-order discrete phase shift Δo_n executed by the system is dictated directly by the local angular frequency:

$$\Delta o_n \approx \omega_o(o_n) \cdot \Delta t_{tick} \quad (85)$$

The complete orbital phase evolution over any measured macroscopic time interval Δt is computationally recovered by iterating the discrete map:

$$\boxed{o_{n+1} = o_n + \left(\frac{\beta \cdot c}{a} \cdot \frac{(1 + e \cdot \cos(o_n))^2}{(1 - e^2)^{\frac{3}{2}}} \right) \Delta t_{tick}} \quad (86)$$

Remark 7.1 (Operational Discretization). *This iterative state machine represents the operational reality of the physical measurement apparatus. The accumulated discretization error within this first-order map physically corresponds to the intra-tick geometric evolution that remains fundamentally unresolvable to an observer bounded by the finite frequency of their local clock. The continuous, transcendental integral is recovered identically only in the unobservable limit $\Delta t_{tick} \rightarrow 0$.*

Epistemological Emergence of the Classical Clock

To prove the closed-form evaluation of ζ_o generates exact classical time evolution without positing an absolute background temporal dimension (dt), we reduce the two terms of the expression independently.

Phase Transformation (Logarithmic Core):

The first term evaluates the phase angle transformation:

$$2 \arctan \left(\frac{\sin(o/2)}{\sqrt{e_X} \cos(o/2)} \right) \quad (87)$$

Substituting the shape factor definition $e_X = \frac{1+e}{1-e}$ yields $\frac{1}{\sqrt{e_X}} = \sqrt{\frac{1-e}{1+e}}$. Recognizing $\tan(o/2) = \frac{\sin(o/2)}{\cos(o/2)}$, the term collapses to:

$$2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \left(\frac{o}{2} \right) \right) \quad (88)$$

This expression is the exact geometric definition of the Eccentric Anomaly (E) mapped from the True Anomaly (o). The first term is identically E .

Rational Trigonometric Projection:

The second term evaluates the internal structural geometry:

$$\frac{e e_Y \sin(o)}{1 + e \cos(o)} \quad (89)$$

Substituting the complementary eccentricity $e_Y = \sqrt{1 - e^2}$ applies the geometric identity mapping true anomaly to eccentric anomaly:

$$\frac{\sqrt{1 - e^2} \sin(o)}{1 + e \cos(o)} = \sin(E) \quad (90)$$

Multiplying by the closure defect e yields exactly the second term.

Reduction:

Substituting the reduced terms back into the primary evaluation yields Kepler's Equation:

$$\zeta_o = E - e \sin(E) \quad (91)$$

The R.O.M. phase integral synthesizes the passage of classical time purely from static S^1 and S^2 relational parameters. Mechanistic orbital time is strictly the linear expansion of the internal temporal phase interval ζ_o required to satisfy the S^1 closure constraint.

Super-Horizon Kinematic Lockout and Imaginary Time

The operational validity of the continuous phase clock is bounded by the relational geometry structural limits. At the kinematic lockout state ($\omega_{pshift} \geq 1$), the S^2 potential carrier reaches saturation ($\kappa^2 \geq 1$). The geometry cannot maintain a bounded, periodic state, forcing the closure defect to break the elliptical limit ($e > 1$).

Evaluating the time integral boundary coefficient under this condition:

$$(1 - e^2)^{\frac{3}{2}} \quad \text{for } e > 1 \quad (92)$$

The term $(1 - e^2)$ evaluates strictly negative. The fractional exponent forces the resulting temporal phase interval ζ_o into the mathematically imaginary domain.

In analytical mechanics, an imaginary temporal interval proves the cessation of periodic physical states. The operational clock ceases to exist because the periodic geometric intersection required to register a relational day is permanently locked out. The trajectory transitions into a pure, non-periodic plunge. This maps identically to the General Relativistic prohibition of stable timelike geodesics inside the event horizon, deriving the plunging state purely from the algebraic shattering of the S^1 bounding limit, eliminating the metric singularity.