

Orbital Mechanics Analytical Reference

Classical and Relativistic Systems

1. Parameter Nomenclature & Origin

Symbol	Definition	SI Units	Formal Origin
μ	Standard Gravitational Parameter (GM)	m^3s^{-2}	Newtonian Gravity
c	Speed of Light in Vacuum	ms^{-1}	Special Relativity
r_s	Schwarzschild Radius ($2\mu/c^2$)	m	General Relativity
a	Semi-major Axis	m	Conic Geometry
e	Eccentricity	-	Conic Geometry
p	Semi-latus Rectum	m	Conic Geometry
n	Mean Motion	rads^{-1}	Keplerian Kinematics
ν, E, M	True, Eccentric, and Mean Anomalies	rad	Keplerian Kinematics
\vec{h}	Specific Angular Momentum Vector	m^2s^{-1}	Newtonian Mechanics
ε	Specific Mechanical Energy	m^2s^{-2}	Newtonian Mechanics
τ	Proper Time	s	General Relativity
\mathcal{E}	Specific Energy (Normalized, $E/(mc^2)$)	-	General Relativity
ℓ	Specific Angular Momentum (Relativistic)	m^2s^{-1}	General Relativity
V_{eff}	Effective Potential	m^2s^{-2}	Analytical Mechanics

2. Classical (Newtonian) Mechanics

2.1. Conservation Laws & Vis-Viva

$$\frac{d\vec{h}}{dt} = 0 \implies \vec{h} = \text{constant}$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$v = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a} \right)}$$

2.2. Trajectory Geometry

$$r(\nu) = \frac{p}{1 + e \cos \nu} = \frac{h^2/\mu}{1 + e \cos \nu}$$

$$p = a(1 - e^2)$$

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$$

$$h = \sqrt{\mu a(1 - e^2)}$$

2.3. Time & Anomalies

$$\begin{aligned}
 n &= \sqrt{\frac{\mu}{a^3}} && \text{Mean Motion} \\
 M &= n(t - t_0) && \text{Mean Anomaly} \\
 M &= E - e \sin E && \text{Kepler's Equation (Elliptical)} \\
 \tan\left(\frac{\nu}{2}\right) &= \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right) && \text{True to Eccentric Anomaly} \\
 T &= 2\pi\sqrt{\frac{a^3}{\mu}} && \text{Orbital Period}
 \end{aligned}$$

2.4. Vector Relations

$$\begin{aligned}
 \vec{e} &= \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r} && \text{Eccentricity Vector (points to periapsis)} \\
 \tan \gamma &= \frac{e \sin \nu}{1 + e \cos \nu} && \text{Flight Path Angle}
 \end{aligned}$$

3. Relativistic (Schwarzschild) Mechanics

3.1. Metric & Proper Time

$$\begin{aligned}
 ds^2 &= -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \\
 c^2 d\tau^2 &= -ds^2 \quad (\text{for timelike geodesics})
 \end{aligned}$$

3.2. Conserved Quantities (Equatorial Plane, $\theta = \pi/2$)

$$\begin{aligned}
 \mathcal{E} &= \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} && \text{Specific Energy (normalized to } c^2) \\
 \ell &= r^2 \frac{d\phi}{d\tau} && \text{Specific Angular Momentum}
 \end{aligned}$$

3.3. Radial Motion & Effective Potential

$$\begin{aligned}
 \left(\frac{dr}{d\tau}\right)^2 &= c^2 \mathcal{E}^2 - \left(1 - \frac{r_s}{r}\right) \left(c^2 + \frac{\ell^2}{r^2}\right) \\
 \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 &= \frac{c^2(\mathcal{E}^2 - 1)}{2} - V_{\text{eff}}(r) \\
 V_{\text{eff}}(r) &= -\frac{\mu}{r} + \frac{\ell^2}{2r^2} - \frac{\mu\ell^2}{c^2 r^3} && \text{Exact Relativistic Effective Potential}
 \end{aligned}$$

3.4. Exact Binet Orbit Equation ($u = 1/r$)

$$\frac{d^2 u}{d\phi^2} + u = \frac{\mu}{\ell^2} + \frac{3\mu}{c^2} u^2$$

3.5. Critical Limits & Geometries

$$\begin{aligned}
 r_{\text{ISCO}} &= \frac{6\mu}{c^2} = 3r_s && \text{Innermost Stable Circular Orbit} \\
 r_{\text{photon}} &= \frac{3\mu}{c^2} = 1.5r_s && \text{Photon Sphere (unstable null geodesic)} \\
 \Delta\phi_{\text{precession}} &= \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{2\ell dr}{r^2 \sqrt{c^2(\mathcal{E}^2 - 1) - 2V_{\text{eff}}(r)}} - 2\pi && \text{Exact Apsidal Precession (per orbit)}
 \end{aligned}$$